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Rate of Convergence of the Linear Discrete Polya Algorithm[1]

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proper affine subspace K of $\mathbb{R}^n$, $h \notin K$, then $\lim_{p \to \infty} h_p = h^*_\infty$, where h^*_∞ is a best uniform approximation of h from K, the so-called strict uniform approximation. Our aim is to give a complete description of the rate of convergence of $\|h_p - h^*_\infty\|$ as $p \to \infty$. © 2001 Academic Press

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1. INTRODUCTION

For $x = (x(1), x(2), ..., x(n)) \in \mathbb{R}^n$, the ℓ_p-norms, $1 \leqslant p \leqslant \infty$, are defined by

$$\|x\|_p = \left(\sum_{j=1}^{n} |x(j)|^p \right)^{1/p}, \qquad 1 \leqslant p < \infty,$$

$$\|x\| := \|x\|_\infty = \max_{1 \leqslant j \leqslant n} |x(j)|.$$

Let $K \neq \varnothing$ be a closed subset of $\mathbb{R}^n$. For $h \in \mathbb{R}^n \backslash K$ and $1 \leqslant p \leqslant \infty$ we say that $h_p \in K$ is a best p-approximation of h from K if

$$\|h_p - h\|_p \leqslant \|f - h\|_p, \qquad \forall f \in K.$$

The existence of h_p is a well known fact. Moreover, if K is in addition a convex set and $1 < p < \infty$, then there is an unique best p-approximation.

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Throughout this paper we assume that K is a proper affine subspace of $\mathbb{R}^n$. In this case, it is known (see for instance [6]) that h_p, $1<p<\infty$, is a best p-approximation of h from K if and only if

$$\sum_{j=1}^{n} (h_p(j)-f(j))\,|h_p(j)-h(j)|^{p-1}\operatorname{sgn}(h_p(j)-h(j))=0, \qquad \forall f\in K. \tag{1}$$

If $p=\infty$ we call h_∞ a best uniform approximation of h from K. In general, the unicity of the best uniform approximation is not guaranteed. However, a unique "strict uniform approximation," h^*_∞, can be defined [3]. The Polya algorithm is an attempt to define h^*_∞ as the limit of the best p-approximation h_p as $p\to\infty$. If K is an affine subspace of $\mathbb{R}^n$, then the Polya algorithm converges to the strict uniform approximation [1, 4, 5],

$$\lim_{p\to\infty} h_p = h^*_\infty.$$

The strict uniform approximation also verifies the next property. Let H denote the set of the best uniform approximations of h from K. For every $g\in H$ we consider the vector $\tau(g)$ whose coordinates are given by $|g(j)-h(j)|$, $j=1, 2, ..., n$, arranged in decreasing order. Then $\tau(h^*_\infty)$ is the only one that gives a minimal lexicographic ordering.

In [2, 4] it is proved that the convergence of h_p to h^*_∞ occurs at a rate no worse than $1/p$. In [4] the authors give a necessary and sufficient condition on K for h_p to coincide with h^*_∞ for p large, and also a necessary and sufficient condition for

$$p\,\|h_p-h^*_\infty\|\to 0 \qquad \text{as} \quad p\to\infty. \tag{2}$$

The aim of this paper is to give a detailed description of the rate of convergence of the Polya algorithm; more precisely, we prove that if (2) holds then there is a number $0<a<1$ such that $p\,\|h_p-h^*_\infty\|/a^p$ is bounded.

2. NOTATION AND PRELIMINARY RESULTS

Without loss of generality we may assume that $h=0$ and $h^*_\infty(j)\geqslant 0$, $1\leqslant j\leqslant n$, and that the coordinates of h^*_∞ are in decreasing ordering. Let $1=\rho_1>\rho_2>\cdots>\rho_s\geqslant 0$ denote all the different values of $h^*_\infty(j)$, $1\leqslant j\leqslant n$, and $\{J_l\}_{l=1}^{s}$ the partition of $J:=\{1, 2, ..., n\}$ defined by $J_l:=\{j\in J : h^*_\infty(j)=\rho_l\}$, $1\leqslant l\leqslant s$. We henceforth put $r=s$ if $\rho_s>0$ and $r=s-1$ if $\rho_s=0$.

We can write $K = h^*_\infty + \mathscr{V}$, where $\mathscr{V}$ is a proper linear subspace of $\mathbb{R}^n$. Note that it is possible to choose a basis $\mathscr{B} = \{v_1, v_2, ..., v_m\}$ of $\mathscr{V}$ and a partition $\{I_k\}_{k=1}^s$ of $I := \{1, 2, ..., m\}$ such that for all $i \in I_k$, $1 \leqslant k \leqslant s$,

(p1) $v_i(j) = 0$, $\forall j \in J_l$, $1 \leqslant l < k$,

(p2) $v_i(j) \neq 0$ for some $j \in J_k$.

Denote $n_l = \text{card}(J_l)$ and $m_k = \text{card}(I_k)$. We have $m_k < n_k$; otherwise we can take a linear combination of the vectors v_i, $i \in I_k$, in such way that the definition of h^*_∞ is contradicted. In this partition it is possible that $I_k = \varnothing$ for some k, $1 \leqslant k \leqslant s$. However, for simplicity of notation, we suppose that $I_k \neq \varnothing$ for $1 \leqslant k \leqslant r$, this involves no loss of generality. In order to get our main theorem, we use the following result [4].

THEOREM 1. *In the above conditions, $p \, \|h_p - h^*_\infty\| \to 0$ as $p \to \infty$ if and only if*

$$\sum_{j \in J_k} v_i(j) = 0, \qquad \forall i \in I_k, \quad 1 \leqslant k \leqslant r.$$

Moreover, $h_p = h^*_\infty$ for p large if and only if

$$\sum_{j \in J_l} v_i(j) = 0,$$

for all $i \in I_k$, $1 \leqslant k \leqslant r$, and for all $1 \leqslant l \leqslant r$.

The following notation will be used throughout the paper. For k, $l = 1, 2, ..., r$, we consider the matrices $M_{kl} = (v_i(j))_{(i,\, j) \in I_k \times J_l}$, and we put $I_0 = \bigcup_{k=1}^r I_k$, $m_0 = \text{card}(I_0)$, $J_0 = \bigcup_{l=1}^r J_l$, $n_0 = \text{card}(J_0)$. Finally, if A is a matrix then we denote by A^T the transpose matrix of A and by $\|A\|$ the row-sum norm of A.

LEMMA 1. *Let $\{x_p\}$ be a sequence of vectors in $\mathbb{R}^m \backslash \{0\}$ such that $p \, \|x_p\| \to 0$ as $p \to \infty$. Then, for a fixed vector $b \in \mathbb{R}^m$ and for all $\beta > 0$,*

$$\left(\beta + \sum_{j=1}^m b(j) \, x_p(j)\right)^p = \beta^p + \beta^{p-1} p \sum_{j=1}^m b(j) \, x_p(j) + \beta^{p-2} R(p),$$

where $R(p) = o(p \, \|x_p\|)$.

Proof. The proof follows immediately from the application of Taylor's formula to the function $\varphi(x) = (1 + x)^p$ at $x = 0$. ∎

3. RATE OF CONVERGENCE

THEOREM 2. *Let K be a proper affine subspace of $\mathbb{R}^n$, $0 \notin K$. For $1 < p < \infty$, let h_p denote the best p-approximation of 0 from K and let h_∞^* be the strict uniform approximation. Suppose that $p\,\|h_p - h_\infty^*\| \to 0$ as $p \to \infty$. Then there are L_1, $L_2 > 0$ such that*

$$L_1 a^p \leqslant p\,\|h_p - h_\infty^*\| \leqslant L_2 a^p, \tag{3}$$

where

$$a = \max_{1 \leqslant l,\, k \leqslant r} \left\{ \frac{\rho_l}{\rho_k} : \sum_{j \in J_l} v_i(j) \neq 0 \text{ for some } i \in I_k \right\} \tag{4}$$

and a is assumed to be 0 if $\sum_{j \in J_l} v_i(j) = 0$ for all $i \in I_k$, $1 \leqslant k$, $l \leqslant r$.

Proof. Let $\mathscr{B} = \{v_1, v_2, ..., v_m\}$ and I_k, $1 \leqslant k \leqslant s$, as above. If $h_p = h_\infty^*$ for p large then, by Theorem 1, $a = 0$ and (3) holds. Therefore we assume $h_p \neq h_\infty^*$ for p large. This condition is just equivalent to

$$\max_{1 \leqslant k,\, l \leqslant r} \left\{ \max_{i \in I_k} \left| \sum_{j \in J_l} v_i(j) \right| \right\} > 0. \tag{5}$$

Putting $f = h_p + v_i$, $i \in I_0$, in (1) we have, for p large,

$$\sum_{j \in J_0} v_i(j)\, h_p(j)^{p-1} + \sum_{j \in J_{r+1}} v_i(j)\, |h_p(j)|^{p-1} \operatorname{sgn}(h_p(j)) = 0, \qquad \forall i \in I_0. \tag{6}$$

This non-linear system can be written as

$$MH_p^T + NK_p^T = 0, \tag{7}$$

where M and N are the matrices defined by

$$M = (v_i(j))_{(i,\, j) \in I_0 \times J_0}, \qquad N = (v_i(j))_{(i,\, j) \in I_0 \times J_{r+1}}$$

and H_p, K_p denote the vectors in $\mathbb{R}^{n_0}$ and $\mathbb{R}^{n_{r+1}}$, respectively, whose components are given by

$$H_p(j) = h_p(j)^{p-1}, \qquad 1 \leqslant j \leqslant n_0$$

$$K_p(j) = |h_p(n_0 + j)|^{p-1} \operatorname{sgn}(h_p(n_0 + j)), \qquad 1 \leqslant j \leqslant n_{r+1}.$$

If $r = s$ then NK_p^T is assumed to be the null vector in $\mathbb{R}^{m_0}$.

Let $\lambda_p = (\lambda_p(1), \lambda_p(2), ..., \lambda_p(m))$ be the vector in $\mathbb{R}^m$ such that

$$h_p = h_\infty^* + \sum_{k=1}^{m} \lambda_p(k)\, v_k.$$

Note that the condition $p\, \|h_p - h_\infty^*\| \to 0$ as $p \to \infty$ is equivalent to $p\, \|\lambda_p\| \to 0$ as $p \to \infty$.

If $j \in J_l$, $1 \leqslant l \leqslant r$, then from Lemma 1,

$$\begin{aligned} h_p(j)^{p-1} &= \left(h_\infty^*(j) + \sum_{k=1}^{m} \lambda_p(k)\, v_k(j) \right)^{p-1} \\ &= \rho_l^{p-1} + \rho_l^{p-2}(p-1) \sum_{k=1}^{m_0} \lambda_p(k)\, v_k(j) + \rho_l^{p-3} R_p(j), \end{aligned}$$

with $R_p(j) = o(p\, \|\Lambda_p\|)$, where $\Lambda_p := (\lambda_p(1), ..., \lambda_p(m_0))$. Thus we can express the vector H_p^T like

$$H_p^T = \Delta_{J_0}^{p-1} \Upsilon^T + (p-1)\, \Delta_{J_0}^{p-2} M^T \Lambda_p^T + \Delta_{J_0}^{p-3} R_p^T,$$

where $\Upsilon := (1, 1, ..., 1) \in \mathbb{R}^{n_0}$, $R_p = (R_p(1), ..., R_p(n_0))$ and $\Delta_{J_0} := (\delta_{ij})_{(i,j) \in J_0 \times J_0}$ is the diagonal matrix of order $n_0 \times n_0$ such that $\delta_{jj} = \rho_l$ if $j \in J_l$, $1 \leqslant l \leqslant r$. Substituting in (7) we obtain the system

$$M(\Delta_{J_0}^{p-1} \Upsilon^T + (p-1)\, \Delta_{J_0}^{p-2} M^T \Lambda_p^T + \Delta_{J_0}^{p-3} R_p^T) + N K_p^T = 0. \tag{8}$$

Let $\Delta_{I_0} = (\tilde{\delta}_{ij})_{(i,j) \in I_0 \times I_0}$ be the diagonal matrix of order $m_0 \times m_0$ such that $\tilde{\delta}_{ii} = \rho_k$ if $i \in I_k$, $1 \leqslant k \leqslant r$. Multiplying (8) by $\Delta_{I_0}^{-p+2} := (\Delta_{I_0}^{-1})^{p-2}$ we have

$$\begin{aligned} &(p-1)\, \Delta_{I_0}^{-p+2} M \Delta_{J_0}^{p-2} M^T \Lambda_p^T \\ &\quad = -\Delta_{I_0}^{-p+2} M \Delta_{J_0}^{p-1} \Upsilon^T - \Delta_{I_0}^{-p+2} M \Delta_{J_0}^{p-3} R_p^T - \Delta_{I_0}^{-p+2} N K_p^T. \end{aligned} \tag{9}$$

Observe that the multiplication by $\Delta_{I_0}^{-p+2}$ is equivalent to divide by ρ_k^{p-2} each of the equations in (6) obtained for $i \in I_k$. This operation is justified because $v_i(j) = 0$ for all $j \in J_l$ if $j < k$. Next we study each of the terms in the former system. An easy computation shows that

$$A(p) := \Delta_{I_0}^{-p+2} M \Delta_{J_0}^{p-2} M^T = \begin{pmatrix} A_{11}(p) & A_{12}(p) & \cdots & A_{1r}(p) \\ A_{21}(p) & A_{22}(p) & \cdots & A_{2r}(p) \\ \cdots & \cdots & \cdots & \cdots \\ A_{r1}(p) & A_{r2}(p) & \cdots & A_{rr}(p) \end{pmatrix},$$

where $A_{ij}(p)$, $i, j = 1, 2, ..., r$, is the matrix of order $m_i \times m_j$ given by

$$A_{ij}(p) = \sum_{k=i}^{r} \left(\frac{\rho_k}{\rho_i}\right)^{p-2} M_{ik} M_{jk}^T.$$

Thus,

$$A_{ij} := \lim_{p \to \infty} A_{ij}(p) = M_{ii} M_{ji}^T.$$

Since M_{ji} is the null matrix if $j > i$, we conclude that

$$A := \lim_{p \to \infty} A(p) = \begin{pmatrix} M_{11} M_{11}^T & 0 & \cdots & 0 \\ M_{22} M_{12}^T & M_{22} M_{22}^T & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ M_{rr} M_{1r}^T & M_{rr} M_{2r}^T & \cdots & M_{rr} M_{rr}^T \end{pmatrix}$$

is a triangular matrix by blocks and so

$$\det(A) = \prod_{i=1}^{r} \det(M_{ii} M_{ii}^T) \neq 0.$$

In particular we have proved that the matrix $A(p)$ is non singular for p large.

Analogously, denoting by $B_p = -\Delta_{I_0}^{-p+2} M \Delta_{J_0}^{p-1} \Upsilon^T$ it is easy to check that

$$B_p(i) = -\rho_k \sum_{l=1}^{r} \left(\frac{\rho_l}{\rho_k}\right)^{p-1} \sum_{j \in J_l} v_i(j), \qquad i \in I_k, \quad 1 \leqslant k \leqslant r.$$

Let a defined by (4). From (5) and Theorem 1, we have $0 < a < 1$. Moreover, $b := \lim_{p \to \infty} \|B_p\|/a^p > 0$.

Similarly, writing $C_p = -\Delta_{I_0}^{-p+2} M \Delta_{J_0}^{p-3} R_p^T$ we obtain

$$C_p(i) = -\frac{1}{\rho_k} \sum_{l=1}^{r} \left(\frac{\rho_l}{\rho_k}\right)^{p-3} \sum_{j \in J_l} v_i(j)\, R_p(j), \qquad i \in I_k, \quad 1 \leqslant k \leqslant r.$$

Since $v_i(j) = 0$ for all $(i, j) \in I_k \times J_l$, $k > l$, and $R_p(j) = o(p \|\Delta_p\|)$, it follows that

$$\lim_{p \to \infty} \frac{\|C_p\|}{p \|\Delta_p\|} = 0.$$

Finally, denoting $D_p = -\Delta_{I_0}^{-p+2} N K_p^T$ we get immediately $\lim_{p \to \infty} \|D_p\|/a^p = 0$.

With this new notation the system (9) can be written as

$$(p-1)\, A(p)\, \Lambda_p^T = B_p + C_p + D_p,$$

and so

$$\begin{aligned}(p-1)\, \|\Lambda_p\| &= \|A(p)^{-1}\,(B_p + C_p + D_p)\| \\ &\leqslant \|A(p)^{-1}\|\,(\|B_p\| + \|C_p\| + \|D_p\|).\end{aligned}$$

Therefore,

$$(p-1)\, \|\Lambda_p\| \left(1 - \frac{\|A(p)^{-1}\|\, \|C_p\|}{(p-1)\, \|\Lambda_p\|}\right) \leqslant \|A(p)^{-1}\|\, \|B_p\| + \|A(p)^{-1}\|\, \|D_p\|. \tag{10}$$

Dividing (10) by a^p and taking limits as $p \to \infty$ we have

$$\lim_{p\to\infty} \frac{p\, \|\Lambda_p\|}{a^p} \leqslant \|A^{-1}\|\, b.$$

In similar way,

$$\|B_p\| \leqslant (p-1)\, \|A(p)\|\, \|\Lambda_p\| \left(1 + \frac{\|C_p\|}{(p-1)\, \|A(p)\|\, \|\Lambda_p\|}\right) + \|D_p\|$$

and then

$$\lim_{p\to\infty} \frac{p\, \|\Lambda_p\|}{a^p} \geqslant \frac{b}{\|A\|}.$$

Finally, we conclude that

$$\frac{b}{\|A\|} \leqslant \lim_{p\to\infty} \frac{p\, \|\Lambda_p\|}{a^p} \leqslant b\, \|A^{-1}\|. \tag{11}$$

If $r = s$ or $J_s = \varnothing$ then the proof is complete. In the other case we put

$$\begin{aligned}h_p &= h_\infty^* + \sum_{i=1}^{m} \lambda_p(i)\, v_i \\ &= h_\infty^* + \sum_{i\in I_0} \lambda_p(i)\, v_i + \sum_{i\in I_s} \lambda_p(i)\, v_i = h_\infty^* + u_p + w_p.\end{aligned}$$

By (11), we have actually proved that $p\,\|u_p\|/a^p$ is bounded. Our purpose is to prove that $p\,\|w_p\|/a^p$ is also. Obviously, we need only consider the case $w_p \neq 0$ for p large. Taking $f = h_p + w_p$ in (1) we obtain

$$\sum_{j \in J_s} w_p(j)\,|u_p(j) + w_p(j)|^{p-1}\,\mathrm{sgn}(u_p(j) + w_p(j)) = 0. \tag{12}$$

If $u_p(j) = 0$ for all $j \in J_s$, then by (12) $w_p(j) = 0$ for all $j \in J_s$ and hence $w_p = 0$. Therefore, we can assume that for all $p \geqslant 1$, $u_p(j) \neq 0$ for some $j \in J_s$ and $w_p \neq 0$. Under these conditions, let $\beta = \inf \|u_p\|/\|w_p\|$. To conclude the proof we will prove that $\beta > 0$. Suppose $\beta = 0$. Then there exists a subsequence, $p_k \to \infty$, such that $\|u_{p_k}\|/\|w_{p_k}\| \to 0$. Let $J_s^{(1)}$ be the set of indices in J_s such that

$$\lim_{k \to \infty} |w_{p_k}(j)|/\|w_{p_k}\| \neq 0.$$

Note that $J_s^{(1)} \neq \varnothing$. Multiplying (12) by $2^{p_k - 1}/\|w_{p_k}\|^{p_k}$ we obtain, for k large,

$$\begin{aligned}
0 &= \sum_{j \in J_s^{(1)}} \frac{|w_{p_k}(j)|}{\|w_{p_k}\|} \left| \frac{2u_{p_k}(j)}{\|w_{p_k}\|} + \frac{2w_{p_k}(j)}{\|w_{p_k}\|} \right|^{p_k - 1} \\
&\quad + \sum_{j \in J_s \setminus J_s^{(1)}} \frac{w_{p_k}(j)}{\|w_{p_k}\|} \left| \frac{2u_{p_k}(j)}{\|w_{p_k}\|} + \frac{2w_{p_k}(j)}{\|w_{p_k}\|} \right|^{p-1} \mathrm{sgn}(u_{p_k}(j) + w_{p_k}(j)) \\
&\geqslant \sum_{j \in J_s^{(1)}} \frac{|w_{p_k}(j)|}{\|w_{p_k}\|} \left| \frac{2u_{p_k}(j)}{\|w_{p_k}\|} + \frac{2w_{p_k}(j)}{\|w_{p_k}\|} \right|^{p_k - 1} \\
&\quad - \sum_{j \in J_s \setminus J_s^{(1)}} \frac{|w_{p_k}(j)|}{\|w_{p_k}\|} \left| \frac{2u_{p_k}(j)}{\|w_{p_k}\|} + \frac{2w_{p_k}(j)}{\|w_{p_k}\|} \right|^{p_k - 1}.
\end{aligned}$$

Taking limits as $k \to \infty$ we get a contradiction. ∎

The following corollary summarizes the results in Theorems 1 and 2.

Corollary 1. *Let K be a proper affine subspace of $\mathbb{R}^n$, $0 \notin K$. Write $K = h_\infty^* + \mathscr{V}$, where $\mathscr{V}$ is a proper linear subspace of $\mathbb{R}^n$, and let $\mathscr{B} = \{v_1, v_2, ..., v_m\}$ a basis of $\mathscr{V}$ and $\{I_k\}_{k=1}^{s}$ a partition of $\{1, 2, ..., m\}$ verifying* (p1) *and* (p2). *Then there exist $L_1, L_2 > 0$ such that*

$$L_1 a^p \leqslant p\,\|h_p - h_\infty^*\| \leqslant L_2 a^p, \tag{13}$$

where

$$a = \max_{1 \leqslant l, k \leqslant r} \left\{ \frac{\rho_l}{\rho_k} : \sum_{j \in J_l} v_i(j) \neq 0 \textit{ for some } i \in I_k \right\}$$

and a is assumed to be 0 *if* $\sum_{j \in J_l} v_i(j) = 0$ *for all* $i \in I_k$, $1 \leqslant k, l \leqslant r$.

Proof. It is sufficient to note that $a = 1$ if and only if $\sum_{j \in J_l} v_i(j) \neq 0$ for some $i \in J_l$. In this case, $p \, \|h_p - h_\infty^*\|$ does not converge to 0 as $p \to \infty$. This condition is equivalent to h_p converging to h_∞^* with rate exactly $1/p$. ▮

A NUMERICAL EXAMPLE. Consider $h_\infty^* = (1, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$ and

$$\begin{aligned} \mathscr{V} = \langle & (1, -1, 0, 0, 1, -1, 2, -1, 0, 1, 1), (1, 1, -1, -1, 0, 1, -1, 1, 2, 1, 1), \\ & (0, 0, 0, 0, 0, 0, -1, 2, -1, 2, 0), (0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0), \\ & (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2) \rangle. \end{aligned}$$

First, we construct the matrix

$$\begin{array}{ccccccccccccc}
 & & J_1 & & & J_2 & & & J_3 & & J_4 & & \\
 & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \\
I_1 & 1 & -1 & 0 & 0 & 1 & -1 & 2 & -1 & 0 & 1 & 1 & v_1 \\
 & 1 & 1 & -1 & -1 & 0 & 1 & -1 & 1 & 2 & 1 & 1 & v_2 \\
I_2 & & & & & & & & & & & & . \\
I_3 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 2 & 0 & v_3 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & v_4 \\
I_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & v_5
\end{array}$$

Note that in this example $I_2 = \varnothing$. Since the sum of coordinates in the diagonals blocks, $(I_k J_k)$, $k = 1, 2, 3$, is zero, we deduce that $p \, \|h_p - h_\infty^*\| \to 0$ as $p \to \infty$. Next we construct the matrix $Q = (q_{kl})_{k, l = 1, 2, 3}$, $q_{kl} = \max_{i \in I_k} |\sum_{j \in J_l} v_i(j)|$ (if $k = 2$ we put $q_{kl} = 0$), and the matrix $R = (\rho_l / \rho_k)_{k, l = 1, 2, 3}$,

$$Q = \begin{array}{cccc} & J_1 & J_2 & J_3 \\ I_1 & 0 & \boxed{1} & 2 \\ I_2 & 0 & 0 & 0 \\ I_3 & 0 & 0 & 0 \end{array} \qquad R = \begin{array}{cccc} & J_1 & J_2 & J_3 \\ I_1 & 1 & \boxed{1/2} & 1/3 \\ I_2 & 2 & 1 & 2/3 \\ I_3 & 3 & 3/2 & 1 \end{array}.$$

By definition we obtain $a = 1/2$ and we conclude that the rate of convergence of h_p to h_∞^* is $1/(p 2^p)$.

Remarks. It is possible to obtain additional information for the rate of convergence for each of the blocks of coordinates $h_p(j)$, $j \in J_l$, $1 \leqslant l \leqslant r$, by means of a inductive procedure. More precisely, considering only the equations in (6) for $i \in I_1$ we have

$$\sum_{j \in J_1} v_i(j)\, h_p(j)^{p-1} = -\sum_{j \in J \setminus (J_1 \cup J_s)} v_i(j)\, h_p(j)^{p-1} - \sum_{j \in J_s} v_i(j)\, |h_p(j)|^{p-1} \operatorname{sgn}(h_p(j)). \tag{14}$$

For $j \in J_1$,

$$h_p(j)^{p-1} = \left(1 + \sum_{k \in I_1} \lambda_p(k)\, v_k(j)\right)^{p-1} = 1 + (p-1) \sum_{k \in I_1} \lambda_p(k)\, v_k(j) + R_p(j),$$

with $R_p(j) = o(p\, \|\Lambda_p^{(1)}\|)$, where $\Lambda_p^{(1)} = (\lambda_p(k))_{k \in I_1}$. Substituting in (14) and denoting by $F_p(i)$ the second member in (14) we obtain the system

$$(p-1)\, M_{11} M_{11}^T (\Lambda_p^{(1)})^T + M_{11} (R_p^{(1)})^T = F_p^T,$$

where $R_p^{(1)} = (R_p(j))_{j \in J_1}$. Since $M_{11} M_{11}^T$ is non-singular and $\|F_p\|/\rho_2^p$ is bounded, we conclude that the $p\lambda_p(i)$, $i \in I_1$, converge to 0 at a rate no worse than ρ_2^p. Taking into account this information and applying the same procedure to the equations in (6) for $i \in I_2$ we deduce that the rate of convergence of $p\lambda_p(i)$ to 0, $i \in I_2$, is at least $(\rho_3/\rho_2)^p$. Now, we can reiterate this argument for the others blocks of coordinates. Finally, this first estimation of the rate of convergence can be used to obtain an estimation more precise. The basic idea is to apply the same technique to the equations in (6) for $i \in \bigcup_{i=1}^k I_i$, $k = 1, 2, ..., r$. Note that this inductive procedure supposes, in fact, another strategy to prove Theorem 1.

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